THE SHUFFLE RELATION OF FRACTIONS FROM MULTIPLE ZETA VALUES

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ABSTRACT. Partial fraction methods play an important role in the study of multiple zeta values. One class of such fractions is related to the integral representations of MZVs. We show that this class of fractions has a natural structure of shuffle algebra. This finding conceptualizes the connections among the various methods of stuffle, shuffle and partial fractions in the study of MZVs. This approach also gives an explicit product formula of the fractions.

1. INTRODUCTION

Let k be a positive integer. For positive integers s_i and variables u_i, $1 \leqslant i \leqslant k$, define

$$\mathfrak{f}\left({s_1,\cdots,s_k \atop u_1,\cdots,u_k} \right) := \frac{1}{(u_1+\cdots+u_k)^{s_1}(u_2+\cdots+u_k)^{s_2}\cdots u_k^{s_k}}. \tag{1}$$

In the spacial case when $s_i = 1$, $1 \leqslant i \leqslant k$, such fractions appeared in connection with differential geometry [3, 4] and polylogarithms [6] where their products were shown to satisfy the shuffle relation. For example,

$$\begin{aligned} \frac{1}{u_1}\frac{1}{(v_1+v_2)v_2} &= \frac{1}{(u_1+v_1+v_2)(v_1+v_2)v_2} + \frac{1}{(u_1+v_1+v_2)(u_1+v_2)v_2} \\ &\quad + \frac{1}{(u_1+v_1+v_2)(u_1+v_2)v_2}. \end{aligned} \tag{2}$$

In general, such fractions occur naturally from multiple zeta values which, since their introduction in the early 1990s, have attracted much attention from a wide range of areas in mathematics and mathematical physics [1, 2, 5, 7, 10, 11, 12, 13, 14, 17]. Multiple zeta values (MZVs) are special values of the multi-variable complex functions

$$\zeta(s_1,\cdots,s_k) = \sum_{n_1>\cdots>n_s>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

at positive integers s_i, $1 \leqslant i \leqslant k$ with $s_1 \geqslant 2$. With the change of variable $n_i = u_i+\cdots+u_k$, $1 \leqslant i \leqslant k$, we have the well-known rational fraction representation of multiple zeta values:

$$\zeta(s_1,\cdots,s_k) = \sum_{u_1,\cdots,u_k\geqslant 1} \mathfrak{f}\left({s_1,\cdots,s_k \atop u_1,\cdots,u_k} \right). \tag{3}$$

For this reason, we will call these fractions $\mathfrak{f}\left({s_1,\cdots,s_k \atop u_1,\cdots,u_k} \right)$ the **MZV fractions**. Thus any relation among MZV fractions gives a corresponding relation among MZVs after summing over the indices u_i's. Indeed, many relations among multiple zeta values are obtained by studying relations among MZV fractions. The method, the so called partial fractional method, can be traced back to Euler in the case when $k = 2$ and remains one of the most effective methods until today [5, 8, 15].

For example, Granville [8] used this method to give a proof of the sum formula. Gangl, Kaneko and Zagier [5] used the fraction formula

$$\frac{1}{m^i}\,\frac{1}{n^j} = \sum_{r+s=i+j}\left(\binom{r-1}{i-1}\frac{1}{(m+n)^r n^s} + \binom{r-1}{j-1}\frac{1}{(m+n)^r m^s}\right), \tag{4}$$

to obtain the well-known Euler's decomposition formula

$$\zeta(i)\zeta(j) = \sum_{r+s=i+j}\left(\binom{r-1}{i-1}\zeta(r,s) + \binom{r-1}{j-1}\zeta(r,s)\right), \; i, j \geqslant 2.$$

This formula of Euler has been generalized recently by the authors [9] to a product formula of any two MZVs.

The study of these fractions are interesting on their own right because of their applications outside of MZVs and that they make sense even if $s_1 = 1$ when $\zeta(s_1, \cdots, s_k)$ is no longer defined. For example when $s_i = 1$ for all $1 \leqslant i \leqslant k$, these fractions are shown to multiply according to the shuffle product rule [3, 6] as mentioned above. However, a product formula for two MZV fractions is known only in this case and in the case of Eq. (4). In this paper we will provide a product formula for any two MZV fractions making use of the general double shuffle framework introduced in our previous work [9] which is obtained with motivation from the shuffle relation and quasi-shuffle (stuffle) relation of MZVs. We are able to apply this general framework by showing that the MZV fractions in Eq. (1) have canonical integral representations, in the spirit of the integral representations of MZVs by Konsevich [13]. By the standard summation process for MZVs, we recover the above mentioned generalization of Euler's decomposition formula of MZVs.

As an example of our product formula, we have

$$\begin{aligned}
\frac{1}{u_1^{r_1}}\,\frac{1}{(v_1+v_2)^{s_1}v_2^{s_2}} = & \sum_{\substack{t_1,t_2,t_3\geqslant 1\\ t_1+t_2=r_1+s_1}} \binom{t_1-1}{r_1-1}\frac{1}{(u_1+v_1+v_2)^{t_1}(v_1+v_2)^{t_2}v_2^{s_2}}\\
& + \sum_{\substack{t_1,t_2,t_3\geqslant 1\\ t_1+t_2+t_3\\ =r_1+s_1+s_2}} \left[\binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-t_3}\frac{1}{(u_1+v_1+v_2)^{t_1}(u_1+v_2)^{t_2}v_2^{t_3}}\right.\\
& \left. + \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\frac{1}{(u_1+v_1+v_2)^{t_1}(v_2+u_1)^{t_2}u_1^{t_3}}\right].
\end{aligned} \tag{5}$$

When $r_1 = s_1 = s_2 = 1$, we get Eq. (2). See Theorem 3.2 for the general formula.

In Section 2, we recall our general framework of double shuffle algebras and show that it encodes the shuffle product of MZV fractions (Theorem 2.1) through their integral representations. The explicit product formula of MZV fractions is given in Section 3 where we also give some examples.

2. The algebra of MZV fractions

In this section, we recall the general double shuffle framework in [9] and apply it to give the shuffle product of MZV fractions.

2.1. **Shuffle product of** MZV **fractions.** Let U be a set. Define the set of symbols

$$\widehat{U} := \{\left[\begin{smallmatrix} r \\ u \end{smallmatrix}\right] \mid r \in \mathbb{Z}_{\geqslant 1}, u \in U\}.$$

Let $M(\widehat{U})$ be the free monoid generated by $\widehat{U}$. Define the free abelian group

$$\mathcal{H}(\widehat{U}) := \mathbb{Z}M(\widehat{U}). \tag{6}$$

We will define a product on $\mathcal{H}(\widehat{U})$ by transporting the shuffle product on another algebra.

Define the set of symbols

$$\overline{U} = \{x_0\} \sqcup \{x_u \mid u \in U\}$$

and let $M(\overline{U})$ be the free monoid on $\overline{U}$. As usual [9, 16], define the shuffle algebra on $\overline{U}$ to be the vector space

$$\mathcal{H}^{ш}(\overline{U}) := \mathbb{Z}M(\overline{U})$$

equipped with the shuffle product ш, namely

$$(\alpha_1\vec{\alpha}') ш (\beta_1\vec{\beta}') = \alpha_1(\vec{\alpha}' ш (\beta_1\vec{\beta}')) + \beta_1((\alpha_1\vec{\alpha}') ш \vec{\beta}'), \quad \alpha_1, \beta_1 \in \overline{U}, \vec{\alpha}', \vec{\beta}' \in M(\overline{U}),$$

with the initial condition $1 ш \vec{\alpha} = \vec{\alpha} = \vec{\alpha} ш 1$.

Define the subalgebra

$$\mathcal{H}_1^{ш}(\overline{U}) = \mathbb{Z} \oplus (\oplus_{u\in U}\mathcal{H}^{ш}(\overline{U})x_u).$$

Define a linear bijection

$$\rho : \mathcal{H}_1^{ш}(\overline{U}) \to \mathcal{H}(\widehat{U}), \quad x_0^{r_1-1}x_{u_1}\cdots x_0^{r_k-1}x_{u_k} \mapsto \left[\begin{smallmatrix} r_1,\cdots,r_k \\ u_1,\cdots,u_k \end{smallmatrix}\right]. \tag{7}$$

We then transport the shuffle product ш on $\mathcal{H}^{ш}(\overline{U})$ to a product $ш_\rho$ on $\mathcal{H}(\widehat{U})$ via ρ, namely

$$\alpha \, ш_\rho \, \beta = \rho(\rho^{-1}(\alpha) ш \rho^{-1}(\beta)). \tag{8}$$

Let $\mathcal{H}^{ш_\rho}(\widehat{U})$ denote the resulting algebra $(\mathcal{H}(\widehat{U}), ш_\rho)$.

Now let U be a set of variables and let $\mathbb{Z}(U)$ be the field of rational functions in U. In other words, $\mathbb{Z}(U)$ is the field of fractions of $\mathbb{Z}[U]$. Consider the $\mathbb{Z}$-submodule

$$\mathbf{PF}(U) := \mathbb{Z}\{\mathfrak{f}\left(\begin{smallmatrix} s_1,\cdots,s_k \\ u_1,\cdots,u_k \end{smallmatrix}\right) \mid s_i \geqslant 1, u_i \in U, 1 \leqslant i \leqslant k, k \geqslant 0\}, \tag{9}$$

where $\mathfrak{f}\left(\begin{smallmatrix} s_1,\cdots,s_k \\ u_1,\cdots,u_k \end{smallmatrix}\right)$ is defined in Eq. (1). The main result of this section is the following

Theorem 2.1. *If U be a set of variables, then the $\mathbb{Z}$-linear map*

$$\mathcal{F} : \mathcal{H}(\widehat{U}) \to \mathbb{Z}(U), \quad \mathcal{F}\left[\begin{smallmatrix} \vec{s} \\ \vec{u} \end{smallmatrix}\right] = \mathfrak{f}\left(\begin{smallmatrix} \vec{s} \\ \vec{u} \end{smallmatrix}\right), \ \mathcal{F}(1) = 1$$

is a $\mathbb{Z}$-algebra homomorphism. In particular, the $\mathbb{Z}$-submodule $\mathbf{PF}(U)$ *of $\mathbb{Z}(U)$, as the image of $\mathcal{F}$, is a $\mathbb{Z}$-subalgebra of $\mathbb{Z}(U)$.*

The proof of this theorem will be given in Section 2.2. We first give a consequence of the theorem.

Corollary 2.2. *The multiplication of two* MZV *fractions in* $\mathbf{PF}(U)$ *satisfies the shuffle relation:*

$$\mathfrak{f}\left(\begin{smallmatrix}\vec{r}\\ \vec{u}\end{smallmatrix}\right)\mathfrak{f}\left(\begin{smallmatrix}\vec{s}\\ \vec{v}\end{smallmatrix}\right)=\mathfrak{f}\left(\left(\begin{smallmatrix}\vec{r}\\ \vec{u}\end{smallmatrix}\right)ш_{\rho}\left(\begin{smallmatrix}\vec{s}\\ \vec{v}\end{smallmatrix}\right)\right). \tag{10}$$

Here $ш_{\rho}$ *is as defined in Eq. (8).*

2.2. **Integral representations of** MZV **fractions.** In Section 2.2.1 we give integral representations of MZV fractions. We then use this integral representation to prove Theorem 2.1.

2.2.1. *Integral representation of* MZV *fractions.* In preparation of our proof of Theorem 2.1, we present an integral representation of MZV fractions which is essentially the same as the well-known integral representation of MZVs by Konsevich [13]. For the sake of being self-contained and for later reference, we provide the notations and some details.

Define

$$A:=\mathbb{R}\{e^{bt}\mid b\geqslant 0\},\quad A^{+}=\mathbb{R}\{e^{bt}\mid b>0\}. \tag{11}$$

Then A and A^{+} are closed under function multiplication and $A=\mathbb{R}\oplus A^{+}$. We define the operator

$$I_0: A^{+}\to A,\quad f(t)\mapsto\int_{-\infty}^{t}f(t_1)dt_1.$$

For any $\lambda>0$ we define the operator

$$I_\lambda: A\to A,\quad f(t)\mapsto\int_{-\infty}^{t}f(t_1)e^{\lambda t_1}dt_1.$$

Then we have the equations:

$$I_0(e^{bt})=\frac{1}{b}e^{bt},\quad b>0, \tag{12}$$

$$I_\lambda(e^{bt})=\frac{1}{b+\lambda}e^{(b+\lambda)t},\quad b\geqslant 0,\lambda>0. \tag{13}$$

So $I_0(A^{+})\subseteq A^{+}$ and $I_\lambda(A)\subseteq A^{+}$ for $\lambda>0$. By a direct computation using Eq. (12) and (13) we obtain

$$I_{\lambda_1}(h_1)I_{\lambda_2}(h_2)=I_{\lambda_1}(h_1I_{\lambda_2}(h_2))+I_{\lambda_2}(I_{\lambda_1}(h_1)h_2), \tag{14}$$

where $\lambda_1,\lambda_2\in\mathbb{R}_{\geqslant 0}$, h_1 is in the domain of I_{λ_1} and h_2 is in the domain of I_{λ_2},

Proposition 2.3. *For any* $\mathfrak{f}\left(\begin{smallmatrix}s_1,\cdots,s_k\\ b_1,\cdots,b_k\end{smallmatrix}\right)\in\mathbb{R}$*, we have the integral representation*

$$\mathfrak{f}\left(\begin{smallmatrix}s_1,\cdots,s_k\\ b_1,\cdots,b_k\end{smallmatrix}\right)e^{(b_1+\cdots+b_k)t}=(I_0^{\circ(s_1-1)}\circ I_{b_1}\circ\cdots\circ I_0^{\circ(s_k-1)}\circ I_{b_k})(1), \tag{15}$$

where $1:\mathbb{R}\to\mathbb{R}$ *is the constant function. In particular,*

$$\mathfrak{f}\left(\begin{smallmatrix}s_1,\cdots,s_k\\ b_1,\cdots,b_k\end{smallmatrix}\right)=(I_0^{\circ(s_1-1)}\circ I_{b_1}\circ\cdots\circ I_0^{\circ(s_k-1)}\circ I_{b_k})(1)\Big|_{t=0}. \tag{16}$$

Proof. We only need to prove Eq. (15) for which we use the induction on $|\vec{s}| = s_1 + \cdots + s_k$. If $|\vec{s}| = 1$, then $k = 1$ and $s_1 = 1$. By Eq. (13) the right hand side of Eq. (15) is $I_{b_1}(1) = \frac{e^{b_1 t}}{u_1}$, which is equal to the left hand side. Let n be a positive integer $\geqslant 2$. Assume that Eq. (15) holds for any $\vec{s}$ with $|\vec{s}| < n$. Now assume that $|\vec{s}| = n$. If $s_1 = 1$, then $k \geqslant 2$. In this case by the induction hypothesis and Eq. (13) the right hand side of Eq. (15) is equal to

$$I_{b_1}(\mathfrak{f}\left(\begin{smallmatrix} s_2,\cdots,s_k \\ b_2,\cdots,b_k \end{smallmatrix}\right)e^{(b_2+\cdots+b_k)t}) = \mathfrak{f}\left(\begin{smallmatrix} s_2,\cdots,s_k \\ b_2,\cdots,b_k \end{smallmatrix}\right)I_{b_1}(e^{(b_2+\cdots+b_k)t}) = \frac{1}{b_1+\cdots+b_k}\mathfrak{f}\left(\begin{smallmatrix} s_2,\cdots,s_k \\ b_2,\cdots,b_k \end{smallmatrix}\right)e^{(b_1+\cdots+b_k)t}$$

which coincides with the left hand side. The argument for $s_1 > 1$ is similar by using Eq. (12) instead of Eq. (13). $\square$

2.2.2. *The proof of Theorem 2.1.* We now take $U = \mathbb{R}_+$ in Section 2.1 and define the set $\widehat{\mathbb{R}_+}$ and the algebra $\mathcal{H}(\widehat{\mathbb{R}_+})$.

Proposition 2.4. *The $\mathbb{R}$-linear map*

$$\Theta : \mathbb{R}\otimes_{\mathbb{Z}}\mathcal{H}(\widehat{\mathbb{R}_+}) \to A, \quad \left[\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right] \mapsto \mathfrak{f}\left(\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right)e^{(b_1+\cdots+b_k)t},\ 1 \mapsto 1$$

is an $\mathbb{R}$-algebra homomorphism.

Proof. Define

$$P_0 : \mathcal{H}^+(\widehat{\mathbb{R}_+}) \to \mathcal{H}(\widehat{\mathbb{R}_+}), \quad P_0(\left[\begin{smallmatrix} s_1,s_2,\cdots,s_k \\ b_1,b_2,\cdots,b_k \end{smallmatrix}\right]) = \left[\begin{smallmatrix} s_1+1,s_2,\cdots,s_k \\ b_1,b_2,\cdots,b_k \end{smallmatrix}\right],$$

$$P_b : \mathcal{H}(\widehat{\mathbb{R}_+}) \to \mathcal{H}(\widehat{\mathbb{R}_+}), \quad P_b(\left[\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 1,s_1,\cdots,s_k \\ b,b_1,\cdots,b_k \end{smallmatrix}\right], \quad P_b(1) = \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right]$$

and take their scalar extensions to $\mathbb{R}$. We show that

$$\Theta \circ P_b = I_b \circ \Theta, \quad b \geqslant 0. \tag{17}$$

For $b = 0$ we have

$$\begin{aligned}\Theta \circ P_0(\left[\begin{smallmatrix} s_1,s_2\cdots,s_k \\ b_1,b_2\cdots,b_k \end{smallmatrix}\right]) &= \Theta(\left[\begin{smallmatrix} s_1+1,s_2,\cdots,s_k \\ b_1,b_2,\cdots,b_k \end{smallmatrix}\right]) = \mathfrak{f}\left(\begin{smallmatrix} s_1+1,s_2,\cdots,s_k \\ b_1,b_2,\cdots,b_k \end{smallmatrix}\right)e^{(1+s_1+\cdots+s_k)t} \\ &= I_0\Big((I_0^{\circ s_1-1}\circ I_{b_1}\circ\cdots\circ I_0^{\circ(s_k-1)}\circ I_{b_k})(1)\Big) = I_0(\Theta(\left[\begin{smallmatrix} s_1,s_2\cdots,s_k \\ b_1,b_2\cdots,b_k \end{smallmatrix}\right])),\end{aligned}$$

where we have used Eq. (15) in the last two equations. The argument for $b > 0$ is similar.

From [9, Proposition 4.3] we obtain

$$P_a(\xi_1)\,ш_\rho\,P_b(\xi_2) = P_a(\xi_1\,ш_\rho\,P_b(\xi_2)) + P_b(P_a(\xi_1)\,ш_\rho\,\xi_2), \tag{18}$$

where ξ_1 is in the domain of P_a and ξ_2 is in the domain of P_b. Now we prove that

$$\Theta(\xi_1\,ш_\rho\,\xi_2) = \Theta(\xi_1)\Theta(\xi_2)$$

for ξ_1, ξ_2 in the free monoid $M(\widehat{\mathbb{R}_+})$ generated by $\widehat{\mathbb{R}_+}$. This is done by induction on $|\xi_1| + |\xi_2|$. Here

$$|\xi_1| = \begin{cases} 1 & \text{if } \xi_1 = 1, \\ r_1+\cdots+r_\ell, & \text{if } \xi_1 = \left[\begin{smallmatrix} r_1,\cdots,r_\ell \\ a_1,\cdots,a_\ell \end{smallmatrix}\right]. \end{cases}$$

If $|\xi_1| = 0$ or $|\xi_2| = 0$, then there is nothing to prove. So we assume that $|\xi_1| \geqslant 1$ and $|\xi_2| \geqslant 1$. Then we can write $\xi_1 = P_a(\xi_1')$ for some $a \in \mathbb{R}_{\geqslant 0}$ and $\xi_1' \in M(\widehat{\mathbb{R}_+})$. Similarly we can write $\xi_2 = P_b(\xi_2')$ for some $b \in \mathbb{R}_{\geqslant 0}$ and $\xi_2' \in M(\widehat{\mathbb{R}_+})$. Then

$$\begin{aligned}
\Theta(\xi_1 ш_\rho \xi_2) &= \Theta(P_a(\xi_1') ш_\rho P_b(\xi_2')) \\
&= \Theta(P_a(\xi_1' ш_\rho P_b(\xi_2'))) + \Theta(P_b(P_a(\xi_1') ш_\rho \xi_2')) \qquad \text{(by Eq. (18)} \\
&= I_a(\Theta(\xi_1' ш_\rho P_b(\xi_2'))) + I_b(\Theta(P_a(\xi_1') ш_\rho \xi_2')) \qquad \text{(by Eq. (17))} \\
&= I_a(\Theta(\xi_1')\Theta(P_b(\xi_2'))) + I_b(\Theta(P_a(\xi_1'))\Theta(\xi_2')) \qquad \text{(by induction assumption)} \\
&= I_a(\Theta(\xi_1')I_b(\Theta(\xi_2'))) + I_b(I_a(\Theta(\xi_1'))\Theta(\xi_2')) \qquad \text{(by Eq. (17))} \\
&= I_a(\Theta(\xi_1'))I_b(\Theta(\xi_2')) \qquad \text{(by Eq. (14))} \\
&= \Theta(P_a(\xi_1'))\Theta(P_b(\xi_2')) \qquad \text{(by Eq. (17)).}
\end{aligned}$$

This completes the induction. □

Taking $t = 0$ in Proposition 2.4, we obtain

Corollary 2.5. *The $\mathbb{R}$-linear map*

$$\Theta : \mathcal{H}(\widehat{\mathbb{R}_+}) \to \mathbb{R}, \quad \left[\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right] \mapsto \mathfrak{f}\left(\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right),\ 1 \mapsto 1$$

is an $\mathbb{R}$-algebra homomorphism.

Based on this corollary we can now prove Theorem 2.1.

Proof of Theorem 2.1. Let $\left[\begin{smallmatrix} \vec{r} \\ \vec{v} \end{smallmatrix}\right] \in \widehat{U}^k$ and $\left[\begin{smallmatrix} \vec{s} \\ \vec{u} \end{smallmatrix}\right] \in \widehat{U}^\ell$. We have to prove the equation

$$\mathcal{F}(\left[\begin{smallmatrix} \vec{r} \\ \vec{v} \end{smallmatrix}\right])\mathcal{F}(\left[\begin{smallmatrix} \vec{s} \\ \vec{u} \end{smallmatrix}\right]) = \mathcal{F}(\left[\begin{smallmatrix} \vec{r} \\ \vec{v} \end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix} \vec{s} \\ \vec{u} \end{smallmatrix}\right]). \tag{19}$$

Both sides of this equation are rational functions in U. Since the zero set of a nonzero rational function does not contain any non-empty open subset in $\mathbb{R}^{k+\ell}$ while, by Corollary 2.5, the above equation holds when the variables $\vec{u}$ and $\vec{v}$ take values in $\mathbb{R}_+$, the equation has been proved. □

3. Product formula of MZV fractions

We now apply Theorem 2.1 and the explicit shuffle product formula obtained in [9] to give an explicit product formula of MZV fractions. We will also give some examples.

We need to recall some notations to give this formula. For positive integers k and ℓ, denote $[k] = \{1, \cdots, k\}$ and $[k+1, k+\ell] = \{k+1, \cdots, k+\ell\}$. Define

$$\mathcal{I}_{k,\ell} = \left\{ (\varphi, \psi) \;\middle|\; \begin{array}{l} \varphi : [k] \to [k+\ell], \psi : [\ell] \to [k+\ell] \text{ are order preserving} \\ \text{injective maps and } \mathrm{im}(\varphi) \cup \mathrm{im}(\psi) = [k+\ell] \end{array} \right\} \tag{20}$$

Let $\vec{u} \in U^k$, $\vec{v} \in U^\ell$ and $(\varphi, \psi) \in \mathcal{I}_{k,\ell}$. We define $\vec{u}\, ш_{(\varphi,\psi)} \vec{v}$ to be the vector whose ith component is

$$(\vec{u}\, ш_{(\varphi,\psi)} \vec{v})_i := \begin{cases} u_j & \text{if } i = \varphi(j), \\ v_j & \text{if } i = \psi(j), \end{cases} \qquad 1 \leqslant i \leqslant k+\ell. \tag{21}$$

Let $\vec{r}=(r_1,\cdots,r_k)\in\mathbb{Z}_{\geqslant 1}^k$, $\vec{s}=(s_1,\cdots,s_\ell)\in\mathbb{Z}_{\geqslant 1}^\ell$ and $\vec{t}=(t_1,\cdots,t_{k+\ell})\in\mathbb{Z}_{\geqslant 1}^{k+\ell}$ with $|\vec{r}|+|\vec{s}|=|\vec{t}|$. Here $|\vec{r}|=r_1+\cdots+r_k$ and similarly for $|\vec{s}|$ and $|\vec{t}|$. Denote $R_i=r_1+\cdots+r_i$ for $i\in[k]$, $S_i=s_1+\cdots+s_i$ for $i\in[\ell]$ and $T_i=t_1+\cdots+t_i$ for $i\in[k+\ell]$. For $i\in[k+\ell]$, define

$$h_{(\varphi,\psi),i}=h_{(\varphi,\psi),(\vec{r},\vec{s}),i}=\begin{cases} r_j & \text{if } i=\varphi(j)\\ s_j & \text{if } i=\psi(j)\end{cases}=r_{\varphi^{-1}(i)}s_{\psi^{-1}(i)}, \tag{22}$$

with the convention that $r_\emptyset=s_\emptyset=1$.

With these notations, we define

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)=\begin{cases}\binom{t_i-1}{h_{(\varphi,\psi),i}-1} & \text{if } i=1 \text{ or if } i-1,i\in\operatorname{im}(\varphi) \text{ or if } i-1,i\in\operatorname{im}(\psi),\\ \binom{t_i-1}{T_i-R_{|\varphi^{-1}([i])|}-S_{|\psi^{-1}([i])|}} \\ =\binom{t_i-1}{\sum_{j=1}^{i}t_j-\sum_{j=1}^{i}h_{(\varphi,\psi),j}} & \text{otherwise.}\end{cases} \tag{23}$$

The following theorem is proved in [9].

Theorem 3.1. **(**[9, Theorem 2.1] *Let U be a countably infinite set and let $\mathcal{H}^{ш_\rho}(\widehat{U})=(\mathcal{H}(\widehat{U}),ш_\rho)$ be as defined by Eq. (8). Then for $\left[\begin{smallmatrix}\vec{r}\\ \vec{u}\end{smallmatrix}\right]\in\widehat{U}^k$ and $\left[\begin{smallmatrix}\vec{s}\\ \vec{v}\end{smallmatrix}\right]\in\widehat{U}^\ell$ in $\mathcal{H}^{ш_\rho}(\widehat{U})$, we have*

$$\left[\begin{matrix}\vec{r}\\ \vec{u}\end{matrix}\right]ш_\rho\left[\begin{matrix}\vec{s}\\ \vec{v}\end{matrix}\right]=\sum_{\substack{(\varphi,\psi)\in\mathcal{I}_{k,\ell}\\ \vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},\,|\vec{t}|=|\vec{r}|+|\vec{s}|}}\left(\prod_{i=1}^{k+\ell}c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\right)\left[\begin{matrix}\vec{t}\\ \vec{u}ш_{(\varphi,\psi)}\vec{v}\end{matrix}\right], \tag{24}$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (23) and $\vec{u}ш_{(\varphi,\psi)}\vec{v}$ is given in Eq. (21).

Then by Corollary 2.2, we have

Theorem 3.2. *With notations as in Theorem 3.1, we have*

$$\mathfrak{f}\left(\begin{matrix}\vec{r}\\ \vec{u}\end{matrix}\right)\mathfrak{f}\left[\begin{matrix}\vec{s}\\ \vec{v}\end{matrix}\right])=\sum_{\substack{(\varphi,\psi)\in\mathcal{I}_{k,\ell}\\ \vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},\,|\vec{t}|=|\vec{r}|+|\vec{s}|}}\left(\prod_{i=1}^{k+\ell}c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\right)\mathfrak{f}\left(\begin{matrix}\vec{t}\\ \vec{u}ш_{(\varphi,\psi)}\vec{v}\end{matrix}\right). \tag{25}$$

Assume $r_1,s_1\geqslant 2$. Taking the sum $\sum\limits_{u_1,\cdots,u_k\geqslant 1}\sum\limits_{v_1,\cdots,v_\ell\geqslant 1}$ on both sides of Eq. (25), we obtain the generalization of Euler's decomposition formula of two MZVs in [9, Corollary 2.5].

We give some examples of Theorem 3.2. We will only provide details for the first example and will refer the reader to [9, Section 2.4] for further details on the computations.

1. The case of $k=\ell=1$**.** Then $\vec{r}=r_1$ and $\vec{s}=s_1$ are positive integers, and $\vec{u}=u_1$ and $\vec{v}=v_1$ are variables. Let $\vec{t}=(t_1,t_2)\in\mathbb{Z}_{\geqslant 1}^2$ with $t_1+t_2=r_1+s_1$. If $(\varphi,\psi)\in\mathcal{I}_{1,1}$, then either $\varphi(1)=1$ and $\psi(1)=2$, or $\psi(1)=1$ and $\varphi(1)=2$. If $\varphi(1)=1$ and $\psi(1)=2$, then as in [9, Section 2.4], we obtain

$$c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}=c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(1)\,c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(2)=\binom{t_1-1}{r_1-1}.$$

By Eq. (21), we have

$$\vec{u}\,ш_{(\varphi,\psi)}\vec{v} = (u_1, v_1).$$

If $\psi(1) = 1$ and $\varphi(1) = 2$, then we similarly obtain

$$c^{\vec{t},(\varphi,\psi)}_{r_1,s_1} = c^{\vec{t},(\varphi,\psi)}_{r_1,s_1}(1)\, c^{\vec{t},(\varphi,\psi)}_{r_1,s_1}(2) = \binom{t_1-1}{s_1-1}.$$

By Eq. (21), we have $\vec{u}\,ш_{(\varphi,\psi)}\vec{v} = (v_1, u_1)$. Therefore,

$$\mathfrak{f}\left(\begin{smallmatrix} r_1\\ u_1\end{smallmatrix}\right)\mathfrak{f}\left(\begin{smallmatrix} s_1\\ v_1\end{smallmatrix}\right) = \sum_{t_1,t_2\geqslant 1, t_1+t_2=r_1+s_1} \binom{t_1-1}{r_1-1}\mathfrak{f}\left(\begin{smallmatrix} t_1,t_2\\ u_1,v_1\end{smallmatrix}\right) + \sum_{t_1,t_2\geqslant 1, t_1+t_2=r_1+s_1} \binom{t_1-1}{s_1-1}\mathfrak{f}\left(\begin{smallmatrix} t_1,t_2\\ v_1,u_1\end{smallmatrix}\right).$$

That is,

$$\frac{1}{u_1^{r_1}}\,\frac{1}{v_1^{s_1}} = \sum_{t_1,t_2\geqslant 1, t_1+t_2=r_1+s_1} \binom{t_1-1}{r_1-1}\frac{1}{(u_1+v_1)^{t_1}v_1^{t_2}} + \sum_{t_1,t_2\geqslant 1, t_1+t_2=r_1+s_1} \binom{t_1-1}{s_1-1}\frac{1}{(u_1+v_1)^{t_1}u_1^{t_2}}.$$

This coincides with the well-known partial fraction formula [5, Eq. (19)] recalled in Eq. (4).

2. The case of $r = 1, s = 2$. By a similar computation of the coefficients, Eq. (25) becomes Eq. (5).

3. The case of $r = s = 2$. In this case $\left[\begin{smallmatrix} \vec{r}\\ \vec{w}\end{smallmatrix}\right] = \left[\begin{smallmatrix} r_1,r_2\\ w_1,w_2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \vec{s}\\ \vec{z}\end{smallmatrix}\right] = \left[\begin{smallmatrix} s_1,s_2\\ z_1,z_2\end{smallmatrix}\right]$. Let $\vec{t} = (t_1, t_2, t_3, t_4) \in \mathbb{Z}^4_{\geqslant 1}$ with $t_1 + t_2 + t_3 + t_4 = r_1 + r_2 + s_1 + s_2$. Then there are $\binom{4}{2} = 6$ choices of $(\varphi, \psi) \in \mathfrak{J}_{2,2}$. Then from Theorem 3.2, we similarly derive

$$\begin{aligned}
&\mathfrak{f}\left[\begin{smallmatrix} r_1,r_2\\ u_1,u_2\end{smallmatrix}\right]\mathfrak{f}\left[\begin{smallmatrix} s_1,s_2\\ v_1,v_2\end{smallmatrix}\right]\\
&= \sum_{\substack{t_1\geqslant 2,\, t_2,t_3\geqslant 1\\ t_1+t_2+t_3=r_1+r_2+s_1}} \binom{t_1-1}{r_1-1}\binom{t_2-1}{r_2-1}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,s_2\\ u_1,u_2,v_1,v_2\end{smallmatrix}\right] + \sum_{\substack{t_1\geqslant 2,\, t_2,t_3\geqslant 1\\ t_1+t_2+t_3=r_1+s_1+s_2}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,r_2\\ v_1,v_2,u_1,u_2\end{smallmatrix}\right]\\
&\quad + \sum_{\substack{t_1\geqslant 2,\, t_2,t_3,t_4\geqslant 1\\ t_1+t_2+t_3+t_4=\\ r_1+r_2+s_1+s_2}} \Bigg[\binom{t_1-1}{r_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-t_4}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,t_4\\ u_1,v_1,u_2,v_2\end{smallmatrix}\right]\\
&\qquad + \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-t_4}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,t_4\\ v_1,u_1,v_2,u_2\end{smallmatrix}\right]\\
&\qquad + \binom{t_1-1}{r_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-1}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,t_4\\ u_1,v_1,v_2,u_2\end{smallmatrix}\right] + \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-1}\mathfrak{f}\left[\begin{smallmatrix} t_1,t_2,t_3,t_4\\ v_1,u_1,u_2,v_2\end{smallmatrix}\right]\Bigg].
\end{aligned}$$

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